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A sufficient condition for the instability of columnar vortices

By S. LEIBOVICH
Sibley School of Mechanical and Aerospace Engineering, Cornell University, Ithaca, NY 14853

AND K. STEWARTSON
Department of Mathematics, University College London

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The inviscid instability of columnar vortex flows in unbounded domains to three-dimensional perturbations is considered. The undisturbed flows may have axial and swirl velocity components with a general dependence on distance from the swirl axis. The equation governing the disturbance is found to simplify when the azimuthal wavenumber n is large. This permits us to develop the solution in an asymptotic expansion and reveals a class of unstable modes. The asymptotic results are confirmed by comparisons with numerical solutions of the full problem for a specific flow modelling the trailing vortex. It is found that the asymptotic theory predicts the most-unstable wave with reasonable accuracy for values of n as low as 3, and improves rapidly in accuracy as n increases. This study enables us to formulate a sufficient condition for the instability of columnar vortices as follows. Let the vortex have axial velocity $W(r)$, azimuthal velocity $V(r)$, where r is distance from the axis, let Ω be the angular velocity V/r, and let Γ be the circulation rV. Then the flow is unstable if

$$V\frac{d\Omega}{dr}\left[\frac{d\Omega}{dr}\frac{d\Gamma}{dr}+\left(\frac{dW}{dr}\right)^2\right] < 0.$$

1. Introduction

We investigate the instability of incompressible inviscid concentrated vortex flows to infinitesimal three-dimensional disturbances. The method of analysis is novel and we expect that it will be applicable to other classes of flows as well. Our ultimate purpose is to determine the role played by hydrodynamic instabilities in the highly nonlinear phenomena, such as vortex breakdown (see Hall 1972; Leibovich 1978), that are known to occur in concentrated vortex flows. This paper, which deals specifically with unbounded vortex flows, is a step in this programme.

The basic motion is assumed to have cylindrical symmetry, and to have a distribution of vorticity that decays with radial distance from the axis of symmetry. We illustrate our method by applying it to the one-parameter model of the trailing vortex

$$V(r) = \frac{q}{r}(1-e^{-r^2}), \tag{1.1a}$$

$$W(r) = e^{-r^2}, \tag{1.1b}$$

where V is the swirl velocity, W is the axial velocity and r is the radial distance from the symmetry axis, all quantities being expressed in a dimensionless form in a cylindrical coordinate system. The parameter q differentiating members of this family is essentially the maximum pitch angle of the helices on which fluid particles move in the flow (1.1), and, without loss of generality, we take it to be positive. By a suitable Galilean transformation that does not affect stability considerations, (1.1) can represent flows with an axial jet or momentum deficit (wake) near $r = 0$. It has been proposed as a representation for the behaviour of a trailing line vortex far downstream of a wing-tip (Batchelor 1964; Lessen, Singh & Paillet 1974) and there is some experimental support for the proposition (Singh & Uberoi 1976); it is also known to be a good empirical fit to flows upstream of vortex breakdowns in certain experiments (Leibovich 1978; Faler & Leibovich 1977; Garg & Leibovich 1979; Escudier, Bornstein & Zehnder 1980). Although we centre our attention on this particular case, the methods that we develop and use here are general.

The linear stability problem for this model of the trailing vortex (we refer to this model as TV hereafter) has been previously considered by Lessen *et al.* (1974) and their numerical results were extended by Duck & Foster (1980) using a different numerical method. The stability problem for a similar flow (representing Long's vortex (Long 1958, 1961)) has been investigated by Foster & Duck (1982), again using numerical methods. These studies indicate that the most dangerous modes are those with positive azimuthal wavenumber n (as defined in our §2), and that the growth rates, maximized with respect to axial wavenumber, increase with n, at least for the values ($0 \leqslant n \leqslant 6$) computed. This raises the question of whether there is a most-unstable mode at finite n, or whether the growth rate continues to increase monotonically with n.

This fundamental question must be addressed by an asymptotic theory, and that is the central point of this paper (§4). We deal exclusively with positive n (for the reasons stated above). We find a class of unstable disturbances whose maximum growth rate indeed increases monotonically with n and approaches a limit as $n \to \infty$.

Furthermore, our asymptotic theory predicts this maximum growth rate accurately at moderate values of n (the error being about 10 % at $n = 2$, 3 % at $n = 3$, and decreasing rapidly as n increases). The unstable modes found are associated with either one or two critical levels, and when $n \gg 1$ are concentrated in a neighbourhood of a finite value r_0 of r: they may therefore be thought of as *ring modes*. Their structure is different from those wall modes, which have a single critical level near the wall, recently found by Maslowe & Stewartson (1982) for rigidly rotating Poiseuille flow in a pipe. It seems likely that both ring and wall modes can arise in concentrated vortex flows contained in tubes. Furthermore, by the nature of the asymptotic analyses developed by Maslowe & Stewartson (1982) and here, the two sets of modes are independent for large n, and can be treated separately.

In order to substantiate our asymptotic theory, we have supplemented the numerical calculations done by Lessen *et al.* (1974) and by Duck & Foster (1980). These authors do not present data on the real part of the frequency (except for $|n| = 1$), neutral modes, or marginal stability. Our numerical studies, which focus on values of n larger than 3, are the subject of §3. We find that many modes are close together, particularly near neutral conditions, and that this makes numerical computation very difficult. This peculiarity is elucidated by the asymptotic theory.

Our numerical results for the TV model indicate that there is a value of q above which all modes are stabilized. This marginal value is approximately 1·6; this is not inconsistent with Lessen *et al.* (1974), who reported that the marginal value was 'slightly greater than 1·5'. The straightforward asymptotic theory for strongly unstable modes does not hold for values of $q > \surd 2$, but can be extended into this range. This extension, together with our numerical results for marginal stability, will be presented in a further paper dealing with the treatment of neutral modes. Here we only note that the extended asymptotic theory accurately predicts the critical value of q for marginal stability for n as low as 3.

Our problem is formulated in §2. Results of a general nature derivable from a theorem of semicircle type due to Barston (1980), and an improved bound on all growth rates for the system, are also presented there.

In §5 we derive a *sufficient* condition for instability of concentrated vortex flows in unbounded domains to three-dimensional disturbances based upon our asymptotic theory. We find the flow is unstable if

$$2VD\Omega[D(rV)(D\Omega)+(DW)^2] < 0. \tag{1.2}$$

Here r is the radial distance from the symmetry axis, V and W are the azimuthal and axial velocity components of the basic flow, $\Omega = V/r$ is the corresponding angular velocity, and $D(\ )$ indicates differentiation with respect to r. This result is consistent with experimental evidence for flow in tubes (Leibovich 1978). It is worth noting that (1.2) is close in form to a criterion for instability found by Ludwieg (1961) for inviscid swirling flow in a narrow annular gap. Ludwieg's result, however, indicates instability for all q and is therefore not in agreement with observation.

2. Formulation and general results

We suppose an incompressible inviscid fluid filling all space has velocity vector $(0, V(r), W(r))$, where r is the radial coordinate in a cylindrical (r,θ,z) coordinate system, $V(r)$ is the azimuthal and W the axial velocity component. A reversal of the sign of V implies instability (Rayleigh's criterion) if $W \equiv 0$, and very likely if $W \neq 0$ as well, and so we shall assume that $V(r)$ is positive. The perturbation (u', v', w') is assumed to be of the form

$$(u', v', w') = (u(r), v(r), w(r)) \exp[i(\alpha z - n\theta - \omega t)], \tag{2.1a}$$

where ω is a (complex) constant to be found and α is a real constant, which, without loss of generality, we take to be positive; n is an integer (often written in the literature as $-m$). Howard & Gupta (1962) show that the Euler equations for the perturbation velocity may be reduced to the following equation for the amplitude $u(r)$ of the radial disturbance:

$$\gamma^2 DSD_* u - \{\gamma^2 + \gamma n^{-1} a(r) + b(r)\} u = 0, \tag{2.1b}$$

where

$$\gamma = \alpha W - \frac{nV}{r} - \omega, \quad \omega = \omega_r + i\omega_i, \tag{2.1c}$$

$$S = r^2(n^2+\alpha^2 r^2)^{-1}, \tag{2.1d}$$

$$a(r) = nrD[S(D_*(\gamma/r) - 2nr^{-3}V)], \tag{2.1e}$$

$$b(r) = -2\alpha V r^{-2} S[\alpha r D_* V + nDW], \tag{2.1f}$$

$$D(\ ) \equiv \frac{d(\ )}{dr}, \quad D_*(\ ) = \frac{1}{r}\frac{d}{dr}[r(\ )]. \tag{2.1g}$$

The boundary conditions on u are

$$u(0) = 0 \quad (|n| \neq 1), \tag{2.2a}$$

$$Du = 0 \quad (|n| = 1), \tag{2.2b}$$

$$u \to 0 \quad \text{as} \quad r \to \infty. \tag{2.2c}$$

We assume that all quantities appearing in (2.1) and (2.2) are dimensionless, having been normalized with respect to a characteristic lengthscale and a characteristic velocity scale. The example that we treat in detail is the trailing vortex, with velocity components given by (1.1).

2.1. *Results from Barston's semicircle theorem*

Semicircle theorems, originated by Howard (1961), locate all unstable modes within a semicircle in the complex phase-speed ($c \equiv \omega/\alpha$) plane. Howard & Gupta (1962) state a semicircle theorem for the axisymmetric ($n = 0$) version of the present problem. Barston (1980) has proved generalized semicircle theorems which are applicable to our problem. One of Barston's results allows us to state that all unstable modes lie within a semicircle in the complex ω-plane with centre at

$$\omega_0 = \tfrac{1}{2}(a_1 + a_2) \tag{2.3}$$

and radius no larger than

$$\omega_{\mathrm{m}} = [[\tfrac{1}{2}(a_2 - a_1)]^2 + \max \tilde{R}^2(r)]^{\frac{1}{2}}, \tag{2.4a}$$

where

$$a_1 = \min_r [\alpha W(r) - n\Omega(r)], \tag{2.4b}$$

$$a_2 = \max_r [\alpha W(r) - n\Omega(r)], \tag{2.4c}$$

$$\tilde{R}^2 = \{(r\Omega\Omega')^2 + \Omega^2(a_2 - a_1)^2\}^{\frac{1}{2}} - r\Omega\Omega', \tag{2.4d}$$

giving

$$\tilde{R}^2 < \max_r [\Omega(a_2 - a_1) + |2r\Omega\Omega'|], \tag{2.4e}$$

where we have assumed $\Omega \geqslant 0$. Let the subscript M denote the maximum over r, then, (2.4a) and (2.4e) provide a convenient upper bound for ω_{m}:

$$\omega_{\mathrm{m}} \leqslant \tfrac{1}{2}(a_2 - a_1) + \Omega + |(rD\Omega^2)_{\mathrm{M}} - \Omega_{\mathrm{m}}^2|^{\frac{1}{2}}. \tag{2.5}$$

For the trailing vortex, $\max\Omega = q$ and $k = 1$, so (2.5) gives

$$\omega_{\mathrm{m}} < \tfrac{1}{2}(a_2 - a_1) + q$$

in this case. If the radius of the semicircle were $\frac{1}{2}(a_2 - a_1)$, then this result would immediately show that all unstable modes must be associated with at least one critical level where $\gamma_{\mathrm{r}}(r) = 0$. Since $\tilde{R} > 0$, however, the radius of the present bound is larger, and we cannot therefore exclude the possibility of an unstable mode without a critical level. However, in our studies of the trailing vortex, all unstable modes found have critical levels. For a different class of flows, viz Γ and W both linear in r^2, Warren (1978) has shown that unstable modes require at least one critical level.

As $n \rightarrow \infty$, however, $b_1 = a_1/n$ and $b_2 = a_2/n$ are finite and not both zero, and ω/n is confined to a semicircle whose radius is

$$\tfrac{1}{2}(b_2 - b_1) + O(n^{-1}). \tag{2.6}$$

For the trailing-vortex example, the radius is less than $\frac{1}{2}(b_2 - b_1) + q/n$. The semicircle theorem does not enable us to conclude that a critical level exists for any unstable modes as $n \rightarrow \infty$, but it shows at least that $n^{-1}\,\gamma_{\mathrm{r}}(r)$ is $O(n^{-1})$ at some level, and not $O(1)$ as it appears to be formally. This suggests the existence of unstable modes with $\omega_{\mathrm{r}} = O(1)$, and this is confirmed in §4.

Several criteria of the Rayleigh type can be derived, following Howard & Gupta (1962), but none is very informative. Perhaps the most useful is the simplest, which states

$$\omega_{\mathrm{i}} \int_0^\infty \left(\frac{a}{|\gamma|^2} + \frac{2\gamma_r b}{|\gamma|^4}\right) |u|\, r\, dr = 0. \tag{2.7}$$

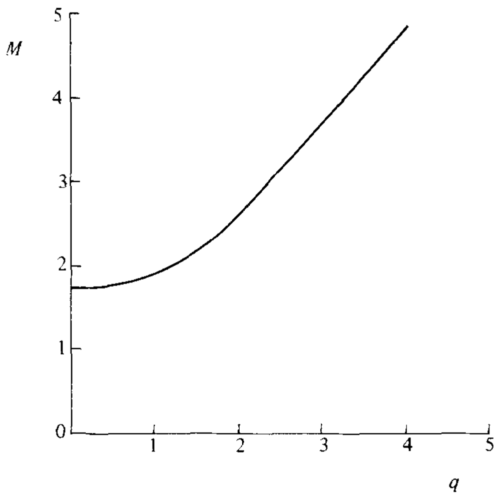

FIGURE 1. Upper bound M for the growth rate of the energy for a disturbance of arbitrary initial amplitude for the trailing-vortex model as a function of the swirl parameter q.

It is easy to show that $a \geqslant 0$ for all positive n, α in the trailing vortex provided $q \geqslant \frac{1}{2}$. Thus, for $q \geqslant \frac{1}{2}$, (2.7) implies that $\gamma_r b$ is negative in some interval in these cases (recall that the cases with $n \geqslant 0$ are the most interesting ones so far as stability is concerned) if the flow is unstable. In particular, there can be no unstable modes if b vanishes identically, as it does for $\alpha q = n$. The explicit formulae for a, b in the TV are given in (4.4) below.

2.2. *An overall bound on the growth rate*

Semicircle theorems give useful bounds on the phase velocities, but not on growth rates. We shall provide a bound on growth rates here.

We consider the growth of any disturbance, whether it be initially infinitesimal or finite, in an inviscid fluid. Assuming only that the disturbance is either periodic or Fourier-transformable in the z-direction, the following equation for the perturbation kinetic energy E holds:

$$\frac{1}{E}\frac{dE}{dt} = -\frac{1}{E}\int_T \mathbf{u}\,.\,\mathbf{S}\,.\,\mathbf{u}\,d\tau \qquad (2.8)$$

(this is a standard starting point of energy-stability theory, see Joseph 1976). Here T is the fluid volume within one period, in the event that periodicity is assumed, or all of space otherwise. In (2·8), $\mathbf{S}$ is the rate-of-strain tensor of the basic blow, and $\mathbf{u}$ is the perturbation velocity vector. In our case

$$\mathbf{S} = \begin{pmatrix} 0 & rD\Omega & DW \\ rD\Omega & 0 & 0 \\ DW & 0 & 0 \end{pmatrix} \qquad (2.9)$$

It is clear from (2.6) that

$$\frac{1}{E}\frac{dE}{dt} \leqslant M, \qquad (2.10)$$

where M is the maximum eigenvalue of $-2\mathbf{S}$. Thus the energy growth rate cannot exceed

$$M = 2\max_r\,[(rD\Omega)^2 + (DW)^2]^{\frac{1}{2}} \qquad (2.11)$$

regardless of the initial amplitude of the perturbation. Figure 1 gives M as a function of q for the trailing vortex.

We now conclude that all unstable modes are located in the complex ω-plane in the intersection of the semicircle defined by (2.4) and the strip $\omega_i < \frac{1}{2}M$ defined by (2.9). Notice that M is independent of wavenumber, and therefore the maximum growth rate is certainly bounded as $n \rightarrow \infty$.

3. Numerical determination of instability for the trailing vortex

We have computed the eigenvalues of (2.1) for the trailing vortex for several values of q and for $n = 1, \ldots, 5$ to supplement the computations of Lessen *et al.* (1974), with our most extensive results for $n = 4$ and 5. Our object is the computation of the dispersion relation, $\omega = \omega_r(\alpha) + i\omega_i(\alpha)$ for the entire band of unstable wavenumbers for fixed values of q and n. Of particular interest is the neutral limit $\omega_i(\alpha) \rightarrow 0$ as $\alpha \rightarrow \alpha_c$; this has not been determined in earlier numerical solutions of this problem (Lessen *et al.* 1974; Duck & Foster 1980).

The general considerations of §2 suggest (but do not prove) that $\gamma_r = 0$ for some value of $r = r_c < \infty$ for unstable modes, and that one should therefore expect a critical layer to exist for neutral modes as well. Although (2.1) fails to have a solution for $\omega_i = 0$, the limit (or limits) $\omega_i(\alpha) \rightarrow 0$ as $\alpha \rightarrow \alpha_c$ can be determined, as is well known, by invoking either viscous effects or by imagining (as we do) that the problem arises in the course of seeking the large-time behaviour of an initial-value problem. To find these singular neutral modes, one then must deform the contour of integration by analytic continuation into the complex r-plane: if γ_r vanishes at $r = r_c$, then the contour must pass above r_c when $D\gamma_r(r_c) < 0$ and below it when $D\gamma_r(r_c) > 0$ (Lin 1955).

Deformation of the contour is the only way that strictly inviscid neutral modes can be computed, but it is also an important numerical expedient. Even for an ω_i that is not evidently 'small', unacceptably slow convergence of a numerical scheme may be encountered if the integration path is not deformed. The limits may also be found by invoking viscous effects, but additional difficulties may then arise (Cotton & Salwen 1981; Maslowe & Stewartson 1982).

Assuming $\gamma(0) \neq 0$, the Frobenius solution of (2.1) that is bounded at the origin is $O(r^{n-1})$ as $r \rightarrow 0$ for $n \geqslant 1$. We solve (2.1) after making a transformation to a new dependent variable w through

$$u = r^{n-2}w(r). \tag{3.1}$$

In this form, w has a finite slope at $r = 0$ that may be taken as unity without loss of generality. The primary solution method that we use is a multiple shooting technique, the SUPORE packaged developed at Sandia by Watts, Scott & Lord (1978) and their coworkers.

We also employ a Galerkin method with Chebychev polynomials as basis functions. This method is satisfactory for unstable modes provided one does not approach neutral wavenumbers. Since it is a matrix method, no initial guess is required, and approximations for the first N eigenvalues are produced. We found the use of an entirely independent method, such as this, to be a valuable check and supplement to the shooting routines. This program, like that of Duck & Foster (1980), shows that there are many unstable modes for each q, α and n. Unfortunately, the advantages of the method are lost when the growth rate of the primary mode (the one with the fastest growth rate) is small. The convergence rates become intolerably slow under these circumstances. In contrast with shooting methods, matrix methods do not isolate the singularity associated with the desired mode but must cope, in effect, with all modes and this, presumably, is the reason that they become less efficient when the growth rates are small. The discussion that follows applies only to the search for the primary mode using shooting procedures.

As $r \to \infty$, (2.1*b*) reduces to

$$D(SD_* u) - u = 0. \tag{3.2}$$

The solution of (3.2) that is bounded at infinity is

$$u(r) = A\,D[K_n(\alpha r)], \tag{3.3}$$

as noted by Lessen *et al.* (1974). The boundary conditions on w are

$$w(0) = 0, \quad Dw(0) = 1, \tag{3.4a}$$

$$w \to 0 \quad \text{as} \quad r \to \infty. \tag{3.4b}$$

Condition (3.4*b*) is implemented in one of two ways. In the first method, we simply set $w(R_2) = 0$, where R_2 was taken to be large enough so that, to the tolerance specified, an increase in R_2 did not affect the computed eigenvalues. Duck & Foster (1980) provide some guidance here, and we found that $R_2 = 4$ was sufficient for the cases $n = 4$ and 5 that we considered. The second method exploits the asymptotic behaviour (3.3). At $r = R$ we set

$$B_1\,Dw(R_2) + B_2\,w(R_2) = 0, \tag{3.5a}$$

where

$$B_1 = R_2^{n-2}[DK_n(\alpha r)]_{r=R_2}, \tag{3.5b}$$

$$B_2 = -D[r^{n-2}DK_n(\alpha r)]_{r=R_2}. \tag{3.5c}$$

This joins the solution to its asymptotic form at $r = R_2$; it is more efficient than setting $w(R_2) = 0$, and reduces computing time by at least 25 %.

In view of the singularity at $r = 0$, the boundary condition (3.4*a*) is enforced by matching to a Frobenius solution in a small interval near $r = 0$. This is conveniently done without modifying the SUPORE package by using the series solution instead of the differential equation to evaluate Dw and D^2w in this small interval.

The geometry of the $\gamma_r(r)$ graph shows that zero, one or two critical points in $r > 0$ are possibilities. Numerical exploration reveals that $\gamma_r = 0$ for at least one point r_{c1} for some unstable wavenumbers, and at two points r_{c1} and $r_{c2} > r_{c1}$ for all others. In fact, wavenumbers having eigenvalues for which $\gamma_r < 0$ at $r = 0$ have one critical point, while those for which $\gamma_r(0) > 0$ have two. We do not know *a priori* where the critical points are located. Consequently, when deformation of the contour is advisable owing to slow convergence of the iterative procedure used, or in the search for neutral modes, a path was chosen that is automatically displaced in the correct direction in the complex r-plane. The choice is

$$r(x) = x\left[1 - i\delta\left(1 - \frac{x}{R_2}\right)D\gamma_r(x)\right]. \tag{3.6}$$

Note that $r(0) = 0$, and $r = R_2$ at $x = R_2$.

The program was checked by comparison with our Galerkin routine, and also with results obtained by Lessen *et al.* (1974) and by Duck (private communication) using Duck & Foster's (1980) program. The cross-checks were all satisfactory: Table 1 shows comparisons with Duck's calculations and with Lessen *et al.* for $n = 4$.

Numerical computations for nearly neutral wavenumbers are very difficult for an unexpected reason. As neutral conditions are approached, it appears that the complex dispersion relations for different modes approach each other. Since they are in close proximity, more than one mode can become involved in the iteration process. Thus

q	α	ω_i D	ω_i LSP	ω_i LS	ω_r D	ω_r LSP	ω_r LS
0·8	0·89	0·0254	—	0·02538	−2·1451	—	−2·14505
—	0·915	0·0349	—	0·03483	−2·1113	—	−2·11124
—	0·920	0·0368	—	0·03684	−2·1046	—	−2·10461
0·82	2·15	—	0·3777	0·37739	—	—	−1·17261

TABLE 1. Comparison with previous computations: D ⟷ Duck, LSP ⟷ Lessen *et al.*, LS ⟷ present work; $n = 4$ in all cases

(presumably) solutions of the differential equation can switch from an approximation to the eigenfunctions of the primary mode, to another approximating the eigenfunction of one of the higher modes. If this occurs, and if there is convergence, then it may be to any of the modes involved in the iteration. Thus, when ω_i is very small for the primary mode, an attempt to find the eigenvalues for a neighbouring wavenumber may yield negative ω_i. This would indicate stability and suggest that the neutral wavenumber has been bracketed, but may, on the contrary, mean only that one has found a higher mode, the primary mode still being unstable.

The asymptotic analysis of §4 helps to explain this odd behaviour, and shows that in a sense things get even worse for large n. For $n \gg 1$, we show in §4 that there are a large number of unstable modes with complex frequencies that coincide to $O(n^{-\frac{1}{2}})$. This suggests that the neutral modes may be more than 'difficult' to compute for large n: unstable modes may be essentially uncomputable by straightforward numerical processes using the differential equation (2.1), and perhaps should be computed using a form for the equation that incorporates the scaling of the asymptotic theory.

The present results confirm the earlier numerical work, which show that the maximum growth rate increases with n for fixed q. Hence the need for a large-n theory is clear. In addition, as the previous paragraph suggests, it is highly likely that results for even moderately large n may be accessible only by means of an asymptotic theory.

Our numerical results, together with those of Lessen *et al.* (1974), will be presented in §4, where a comparison with the asymptotic theory will be made.

4. Analysis for general unstable modes when $n \gg 1$

Problem (2.1) for the eigenvalue ω may be written in the alternative form

$$D^2\phi = K(r;n;\beta;\omega;q)\,\phi. \tag{4.1}$$

Here

$$\beta = \frac{\alpha}{n}, \quad \phi = \left(\frac{r^3}{1+\beta^2r^2}\right)^{\frac{1}{2}}u, \tag{4.2}$$

$$K = n^2\frac{1+\beta^2r^2}{r^2}\left\{1+\frac{a}{n\gamma}+\frac{b}{\gamma^2}-\frac{1+10\beta^2r^2-3\beta^4r^4}{4n^2(1+\beta^2r^2)^3}\right\},$$

$$\gamma = n[\beta W(r)-\Omega(r)]-\omega \equiv n\Lambda(r)-\omega, \tag{4.3a}$$

$$a = rD\left[\frac{1}{1+\beta^2r^2}\frac{1}{r}(\beta r^2+qn)DW\right], \tag{4.3b}$$

$$b = \frac{\beta r^2 \Phi}{q(1+\beta^2 r^2)}(1-\beta q), \tag{4.3c}$$

$$\Phi = r^{-3}\frac{d}{dr}(rV)^2, \quad q = -r\frac{D_* V}{DW},$$

and β is assumed to be $O(1)$.

In the trailing-vortex case to be considered in detail, we take n positive, q a positive constant, and

$$a(r) = \frac{4r^2\, e^{-r^2}}{(1+\beta^2 r^2)^2}[q+\beta^2 q - \beta + (\beta + q\beta^2)\, r^2 + \beta^3 r^4], \tag{4.4a}$$

$$b(r) = \frac{4\beta q(1-\beta q)\, e^{-r^2}(1-e^{-r^2})}{1+\beta^2 r^2}, \tag{4.4b}$$

$$\gamma(r) = n[\beta e^{-r^2} - q r^{-2}(1-e^{-r^2})] - \omega. \tag{4.4c}$$

Notice that, as $r \to \infty$ for the trailing vortex, $a \to 0$, $b \to 0$, and we confine our attention to basic flows of this kind. Further, $a(0) = b(0) = 0$ for the TV (4.4a, b): this holds quite generally for all kinematically possible basic flows. To complete the present reformulation of our problem, note that (2.2) implies boundary conditions

$$\phi(0) = 0, \quad \phi \to 0 \quad \text{as} \quad r \to \infty. \tag{4.5}$$

Provided $\gamma(0) \neq 0$, $K(r;\, n;\, \beta;\, \omega;\, q)$ is real and positive as $r \to \infty$. If the real part of K is positive for all r, there can be no solution of the boundary-value problem. To see this let $K = |K|\, e^{i\nu(r)}$. If $\operatorname{Re} K > 0$ for all r, $\cos \nu > \delta > 0$ for some value of δ. Write $\phi = R(r)\, e^{i\theta(r)}$, substitute into (4.1), multiply by $\exp(-i\theta)$ and take the real part: this yields

$$D^2 R = [|K| \cos \nu + (D\theta)^2]\, R > (\delta|K| + (D\theta)^2)\, R. \tag{4.6a}$$

If $R(0) = 0$, then, by Sturmian theory, R cannot have a second zero.

Thus, for a non-trivial solution to exist, $\operatorname{Re} K$ must vanish at some point $r = r_1$. Furthermore, since $K \to +\infty$ as $r \to 0$ and $K \to n^2$ as $r \to \infty$, $\operatorname{Re} K$ must have an even number of roots on the real r-axis, and we may therefore assume that there is at least a second point $r = r_2$, with $0 < r_1 < r_2$ at which $\operatorname{Re} K$ vanishes. Since

$$1 - \tfrac{1}{4} n^{-2}(1 + 10\beta^2 r^2 - 3\beta^4 r^4)(1+\beta^2 r^2)^{-3} > (1 - 0{\cdot}04 n^{-2}) \quad \text{for all } n > 1,$$

the real part of K can vanish only when

$$\operatorname{Re}\left(\frac{a}{n\gamma} + \frac{b}{\gamma^2}\right) < -(1 - 0{\cdot}24 n^{-2}) \tag{4.6b}$$

in an interval (r_1, r_2).

Now consider the situation as $n \to +\infty$. Since

$$\gamma = n\Lambda - \omega,$$

where Λ, like a and b, is independent of n, and, since ω_i is bounded (from §2), (4.6) requires $\gamma_r = O(1)$ in (r_1, r_2), and hence $\omega_r = O(n)$. For any unstable mode then, as $n \to \infty$ we can neglect $a/n\gamma$ in comparison with b/γ^2, and, in (r_1, r_2), we must have

$$\operatorname{Re}(1 + b\gamma^{-2}) < 0. \tag{4.7}$$

Since Λ is a smooth function of n, the interval $r_2 - r_1$ must be small and vanishes in the limit $n \to \infty$.

The requirement for a negative minimum of Re K, implied by (4.6a) *et seq.*, leads us to search for stationary points of the complex function K. The strategy we adopt to solve the eigenvalue problem is to focus attention upon the neighbourhood of such a stationary point, located at $r = r_0$ with $r_1 < r_0 < r_2$. In this neighbourhood we write

$$K = K_0 + K_2(r-r_0)^2 + K_3(r-r_0)^3 + \ldots, \tag{4.8}$$

where the K_j are independent of r. On the assumption that the K_j are suitably ordered with n, the dominant part of ϕ must satisfy

$$D^2\bar{\phi} = [K_0 + K_2(r-r_0)^2]\,\bar{\phi}, \tag{4.9}$$

and we must have $\bar{\phi} \to 0$ as r leaves the immediate neighbourhood of $r = r_0$ in order to match with the solution of (4.1) elsewhere (this may be represented, as $n \to \infty$, by a WKBJ approximation with no turning points). An exponentially growing ϕ as $|r-r_0|$ increases could not be so matched. Equation (4.9) is satisfied by Weber functions, and the requirement that a solution exists is that

$$K_0(K_2)^{-\frac{1}{2}} = -(2s-1), \tag{4.10}$$

where s is a positive integer. The Weber function corresponding to $s = 1$ is

$$\bar{\phi} = \exp\left[-\tfrac{1}{2}K_2^{\frac{1}{2}}(r-r_0)^2\right], \tag{4.11}$$

and the forms for other values of s follow by differentiation.

We are now in a position to find the condition under which the neglect of $K_j, j > 2$, is justified, and to set up a formal asymptotic expansion for ϕ and ω in descending powers of n. In addition to $n \gg 1$, the only requirement is that ω_i is not small. The neutral and nearly neutral modes for which ω_i is zero or small must be considered separately, and this we shall do in a subsequent paper.

To lowest order in n^{-1},

$$K = n^2\frac{1+\beta^2r^2}{r^2}\left(1+\frac{b}{\gamma^2}\right). \tag{4.12}$$

Since $r_2 \to r_1$ as $n \to \infty$, the stationary point $r = r_0$ to a first approximation is a double zero of K. From (4.12), it is defined by

$$\gamma_0 = -ib_0^{\frac{1}{2}}, \tag{4.13a}$$

$$D\gamma(r_0) = nD\Lambda(r_0) = -\frac{Db(r_0)}{2\gamma_0} = -\tfrac{1}{2}ib_0^{-\frac{1}{2}}Db(r_0), \tag{4.13b}$$

where $\gamma_0 = \gamma(r_0)$ and $b_0 = b(r_0)$. The choice of sign in (4.13a) is made in view of the assumption $\omega_i > 0$. Provided $b_0 \neq 0$, (4.13b) can be replaced by

$$D\Lambda(r_0) = 0. \tag{4.13c}$$

From this point on, we only treat the TV example, but the methodology is clearly general. For the TV, b vanishes only at $r = 0$ or ∞, or when $\beta q = 1$, in which case b vanishes identically and the preceding discussion breaks down altogether. The cases $r = 0$ and $r = \infty$ are not relevant, since we know that in both cases K is real and positive. Thus, excluding the case $\beta q = 1$, we may find r_0 in terms of β and q from (4.13c), which for the TV is

$$\exp r_0^2 = 1 + r_0^2 + \frac{\beta r_0^4}{q}. \tag{4.14}$$

This equation has no real (non-zero) roots if $\beta < \frac{1}{2}q$, and one real positive root if $\beta > \frac{1}{2}q$, and further b vanishes at $\beta q = 1$. Hence our interest is confined to the interval $\frac{1}{2}q < \beta < 1/q$.

Near $r = r_0$,
$$\gamma = \gamma_0 + n\Lambda_2(r-r_0)^2, \tag{4.15a}$$
where
$$\gamma_0 = -n\exp(-r_0^2)[\beta r_0^2 + q - \beta] - \omega, \tag{4.15b}$$
$$\Lambda_2 = 2\exp(-r_0^2)[\beta r_0^2 + q - 2\beta]. \tag{4.15c}$$

In view of (4.13*a*), we know that $1 + b_0/\gamma_0^2$ is $O(1)$. To determine its precise order, we expand K as indicated in (4.8) and find $K_2 = O(n^3)$; (4.10) therefore implies $K_0 = O(n^{\frac{3}{2}})$. The series for γ_0 is therefore expected to proceed in powers of $n^{-\frac{1}{2}}$. Furthermore, the neighbourhood of r_0 in which the truncation in (4.9) is valid is
$$r - r_0 = O(n^{-\frac{3}{4}}).$$

We are now prepared to develop a formal expansion in descending powers of n for the solution of (4.1).

The expansion will satisfy a sequence of differential equations with complex coefficients. These are transformed to a sequence with real coefficients if we extend the problem to the complex r-plane and deform the contour in the direction $\arg(r - r_0) = -\frac{1}{8}\pi$. By this deformation of contour, we do not pass over a zero of γ since (4.13*a*) and (4.15*a*) show that this zero lies either in the first or third quadrant of the complex $(r-r_0)$-plane.

Furthermore, it turns out to be more convenient to expand in inverse powers of $n\exp(-\frac{3}{2}\pi i)$ rather than n. The reason is that if $n = ip$, $\alpha = iA$, where p, A are real and positive, the eigenvalue problem to determine $v = i\omega$ in terms of p, A is essentially real. In particular, for large p the eigenfunctions and eigenvalues may be expanded in series of descending powers of p whose coefficients are real. Thus, proceeding formally, we let
$$p = ne^{\frac{3}{2}\pi i},$$
$$r = r_0 - p^{-\frac{3}{4}}\tau = (r_0 + n^{-\frac{3}{4}}\tau\, e^{-\frac{1}{8}\pi i}), \tag{4.16a}$$
$$\gamma_0 = -i[\Gamma_0 + p^{-\frac{1}{2}}\Gamma_1 + p^{-1}\Gamma_2 + p^{-\frac{3}{2}}\Gamma_3] + O(p^{-2}), \tag{4.16b}$$
where
$$\Gamma_0 = b_0^{\frac{1}{2}}. \tag{4.16c}$$
We define
$$\sigma(r) = \frac{1+\beta^2 r^2}{r^2}, \tag{4.17}$$
and expand the functions a, b and σ, in Taylor series about $r = r_0$. Expressed in terms of τ through (4.16*a*), these may be written in the form
$$a = \sum_{j=0} a_j(-\tau)^j p^{-\frac{3}{4}j}, \tag{4.18a}$$
$$b = \sum_{j=0} b_j(-\tau)^j p^{-\frac{3}{4}j}, \tag{4.18b}$$
$$\sigma = \sum_{j=0} \sigma_j(-\tau)^j p^{-\frac{3}{4}j}, \tag{4.18c}$$
where
$$b_0 = b(r_0), \quad b_j = \frac{1}{j!}\frac{d^j b}{dr^j}(r_0), \tag{4.18d}$$
and the same notation is used for the a- and σ-expansions. Following the same procedure with γ, we have to order $p^{-\frac{3}{2}}$:
$$\gamma = n\Lambda - \omega = \gamma_0 + ip\sum_{j=2} \Lambda_j(-\tau)^j p^{-\frac{3}{4}j}, \tag{4.18e}$$

where now

$$\Lambda_k = \frac{1}{k!}\frac{d^k\Lambda(r_0)}{dr^k}. \tag{4.18f}$$

If (4.16) and (4.18) are substituted into (4.1), we may arrange the result in the form

$$\frac{d^2\phi}{dt^2} - G_0(\tau)\,\phi = \left(\sum_{k=1}^{4} G_k(\tau)\,p^{-\frac{1}{4}k}\right)\phi, \tag{4.19}$$

where the G_k, $k = 0, \ldots, 4$, are polynomials in τ, even or odd if k is even or odd.

The eigenfunction ϕ is now itself expanded in a series in descending integer powers of $p^{-\frac{1}{4}}$:

$$\phi = \phi_0 + \sum_{k=1} p^{-\frac{1}{4}k}\phi_k.$$

This produces a sequence of equations, the first of which is homogeneous, and the succeeding ones are inhomogeneous,

$$\frac{d^2\phi_0}{d\tau^2} - G_0\,\phi_0 = 0, \tag{4.20a}$$

$$\frac{d^2\phi_k}{d\tau^2} - G_0\,\phi_k = \sum_{i=1}^{k} G_i\,\phi_{k-i}, \tag{4.20b}$$

and we must have

$$\phi_j \to 0 \quad \text{as} \quad |\tau| \to \infty. \tag{4.21}$$

The important advantage of these formal expansions in descending powers of p is that all new quantities defined are real. For example, $G_k(\tau)$ is real, Γ_s is real, and γ is purely imaginary (when p is treated as real). The coefficient G_0 is

$$G_0 = \frac{2\sigma_0}{b_0^{\frac{1}{2}}}[\Lambda_2\tau^2 - \Gamma_1], \tag{4.22}$$

and the operator appearing in (4.20) may be put in a cleaner form by scaling τ as follows:

$$\tau = \lambda x, \tag{4.23}$$

with

$$\lambda = \frac{b_0^{\frac{1}{8}}}{(2\Lambda_2\sigma_0)^{\frac{1}{4}}}. \tag{4.24}$$

Equation (4.20a) is then

$$\frac{d^2\phi_0}{dx^2} + \left[\Gamma_1\left(\frac{4\sigma_0^2}{\Lambda_2^2 b_0}\right)^{\frac{1}{4}} - x^2\right]\phi_0 = 0.$$

In order to ensure that ϕ vanishes as $|x| \to \infty$,

$$\Gamma_1 = (2s-1)\left(\frac{\Lambda_2^2 b_0}{4\sigma_0^2}\right)^{\frac{1}{4}} \tag{4.25}$$

for $s = 1, 2, \ldots$ (see (4.10)). When $s = 1$, (4.20a) is

$$\frac{d^2\phi_0}{dx^2} + (1 - x^2)\,\phi_0 = 0. \tag{4.26}$$

Equation (4.26) provides the leading term for the primary unstable mode ($s = 1$), and (4.25) with $s = 1$, together with (4.16b, c) provide an approximation, good to $O(n^{-\frac{1}{2}})$, for the growth rate of unstable modes, i.e.

$$\begin{aligned}\omega_i &= b_0^{\frac{1}{2}} - \sqrt{\tfrac{1}{2}}\,n^{-\frac{1}{2}}\Gamma_1 \\ &= b_0^{\frac{1}{2}} - n^{-\frac{1}{2}}\left(\frac{\Lambda_2^2\, b_0}{16\sigma_0^2}\right)^{\frac{1}{4}}.\end{aligned} \tag{4.27}$$

Since Γ_2 is real, the next term in the expansion of ω_i in inverse powers of $n^{-\frac{1}{2}}$ is $-\Gamma_3\, n^{-\frac{3}{2}}2^{-\frac{1}{2}}$, and we shall now calculate it. First, we observe that the solution to (4.26) is

$$\phi_0 = \exp\left(-\tfrac{1}{2}x^2\right), \tag{4.28}$$

and set

$$\phi_k(x) = \psi_k(x)\exp\left(-\tfrac{1}{2}x^2\right), \quad \psi_0(x) = 1, \tag{4.29}$$

then the ψ_k are determined by

$$\frac{d^2\psi_k}{dx^2} - 2x\frac{d\psi_k}{dx} = \lambda^2 \sum_{j=1}^{k} G_j(x)\,\psi_{k-j}. \tag{4.30}$$

We can express the G_j as follows:

$$G_1 = \lambda g_{11}x, \tag{4.31a}$$

$$G_2 = -[g_{20} + g_{22}x^2 + g_{24}x^4], \tag{4.31b}$$

$$G_3 = -\frac{2\lambda^3}{b_0^{\frac{1}{2}}}[g_{31}x + g_{33}x^3], \tag{4.31c}$$

$$G_4 = -\sigma_0\lambda^2[g_{40} + g_{42}x^2 + g_{44}x^4 + g_{46}x^6], \tag{4.31d}$$

where the g_{mn} are all constants. Up to $m = 3$, these are all known in terms of σ_0, b_0, b_1 and Λ_2, except for g_{20}, which depends upon the unknown Γ_2 as well as these quantities. It is easy to see that an acceptable solution of (4.30) for odd k, $k = 2m+1$ say, must always be of the form

$$\psi_{2m+1} = A_{2m+1}\sum_{j=0}^{2m} c_{2m+1,\,2j+1}\,x^{2j+1}, \tag{4.32a}$$

and for even $k = 2m$ of the form

$$\psi_{2m} = A_{2m}\sum_{j=1}^{2m} c_{2m,\,2j}\,x^{2j}, \tag{4.32b}$$

where the c_{pq} are constants and the A_p are scaling constants chosen for convenience. The ψ_k for odd k, that is, the $c_{2m+1,\,2j+1}$, are determined directly in terms of the coefficients of the relevant equation. By contrast, an acceptable solution can only be obtained for ψ_{2m} when the coefficients g_{mn} satisfy a compatibility condition which requires a linear combination of them to vanish. We show how the procedure works for $k = 1$ and 2. For $k = 1$,

$$\frac{d^2\psi_1}{dx^2} - 2x\frac{d\psi_1}{dx} = \lambda^3 g_{11}x, \tag{4.33a}$$

where

$$g_{11} = -\frac{\sigma_0 b_1}{b_0}. \tag{4.33b}$$

Thus

$$\psi_1 = -\tfrac{1}{2}\lambda^3 g_{11}x, \tag{4.33c}$$

and we take $A_1 = \lambda^3$ and $c_{11} = -\frac{1}{2}g_{11}$. For $k = 2$,

$$\frac{d^2\psi_2}{dx^2} - 2x\frac{d\psi_2}{dx} = -\lambda\left[g_{20} + g_{22}x^2 + g_{24}x^4 - \frac{b_0^{\frac{1}{2}}}{2\sigma_0\Lambda_2}c_{11}g_{11}x^2\right]. \tag{4.34a}$$

Take

$$\psi_2 = -\lambda^2(c_{22}x^2 + c_{24}x^4); \tag{4.34b}$$

then c_{22} and c_{24} must satisfy three equations:

$$2c_{22} = \left(2\Gamma_2 + a_0 - \frac{3\Lambda_2}{2\sigma_0}\right)\frac{\sigma_0}{b_0^{\frac{1}{2}}}, \tag{4.35a}$$

$$8c_{24} = \frac{3\Lambda_2}{2b_0^{\frac{1}{2}}}, \tag{4.35b}$$

$$12c_{24} - 4c_{22} = \frac{3\Lambda_2}{b_0^{\frac{1}{2}}} + \frac{b_1^2\sigma_0}{4b_0^{\frac{3}{2}}\Lambda_2}. \tag{4.35c}$$

The overdetermined set (4.35) has a solution if and only if

$$\Gamma_2 = -\tfrac{1}{2}a_0 + \frac{9\Lambda_2}{16\sigma_0} - \frac{b_1^2}{16\Lambda_2 b_0}. \tag{4.36}$$

Thus (4.36) produces a real solution for Γ_2, which implies that there is no correction to (4.27) to $O(n^{-1})$.

Having found Γ_2, we see that all coefficients g_{mn} are known except for g_{40}, which must be determined by a compatibility condition entirely analogous to (4.35), (4.36). The constant g_{40} contains Γ_3, the $O(n^{-\frac{3}{2}})$ correction to the eigenvalue. The algebra required to get g_{40} and hence Γ_3 is long but straightforward, and we only quote the end result in a schematic form. If we let

$$\psi_3 = -\frac{\lambda^5}{b_0^{\frac{1}{2}}}(c_{31} + c_{33}x^3 + c_{35}x^5),$$

$$\psi_4 = -\frac{\lambda^8}{b_0^{\frac{1}{2}}}(c_{42}x^2 + c_{44}x^4 + c_{46}x^6 + c_{48}x^8),$$

then compatibility for ψ_4 requires

$$\begin{aligned} g_{40} = \frac{1}{32\sigma_0^2\Lambda_2}\Bigg\{ & \frac{210\Lambda_2\sigma_0^2 g_{24}c_{24}}{b_0^{\frac{1}{2}}} \\ & + 30\left[\frac{2\Lambda_2\sigma_0^2}{b^{\frac{1}{2}}}(g_{24}c_{22} + g_{22}c_{24}) - 2\Lambda_2\sigma_0^2 g_{46} - g_{11}c_{35}\right] \\ & + 12\left[\frac{2\Lambda_2\sigma_0^2}{b_0^{\frac{1}{2}}}(g_{20}c_{24} + g_{22}c_{22}) - 2\Lambda_2\sigma_0^2 g_{44} - g_{11}c_{33} - 2g_{33}c_{11}\right] \\ & + 8\left[\frac{2\Lambda_2\sigma_0^2}{b_0^{\frac{1}{2}}}g_{20}c_{22} - 2\Lambda_2\sigma_0^2 g_{42} - g_{11}c_{31} - 2g_{31}c_{11}\right]\Bigg\}, \end{aligned} \tag{4.37}$$

from which Γ_3 is determined by

$$\Gamma_3 = \tfrac{1}{2}\lambda^2 b_0^{\frac{1}{4}}\left\{g_{40} + \frac{11\Lambda_2^2}{8\sigma_0 b_0} - \frac{3}{8}\left(\frac{b_1}{b_0}\right)^2 - \frac{2a_0\Lambda_2}{b_0}\right\}. \tag{4.38}$$

All coefficients entering these formulas are defined in the appendix in terms of a_0, b_0, b_1, b_2, σ_0, σ_1, Λ_2 and Λ_3.

The growth rate to $O(n^{-\frac{3}{2}})$ is

$$\omega_i = b_0^{\frac{1}{2}} - n^{-\frac{1}{2}}\left(\frac{\Lambda^2 b_0}{16\sigma_0}\right)^{\frac{1}{4}} + \surd\tfrac{1}{2}\,\Gamma_3 n^{-\frac{3}{2}} + O(n^{-2}). \tag{4.39}$$

In addition, the real frequency ω_r is found from the real part of (4.15*b*) to be

$$\omega_r = -ne^{-r_0^2}[\beta r_0^2 + q - \beta] + n^{-\frac{1}{2}}\Gamma_2 + \surd\tfrac{1}{2}\,n^{-\frac{3}{2}}\Gamma_3 + O(n^{-2}). \tag{4.40}$$

n	q	$(\omega_{iAS})_{max}$	$(\omega_{iN})_{max}$	β_{AS}	β_N	$(\omega_{iN}-\omega_{iAS})n^2$
1	0·32	0·090	0·147	0·257	0·3	0·057
2	0·7	0·282	0·314	0·470	0·6	0·128
3	0·79	0·343	0·354	0·504	0·57	0·099
4	0·82	0·372	0·378	0·513	0·54	0·096
5	0·83	0·389	0·391	0·514	0·52	0·050
6	0·83	0·399	0·401	0·511	0·53	0·072

TABLE 2. Comparison of maximum growth rate $(\omega_{iAS})_{max}$ calculated from the asymptotic theory holding n and q fixed, and the maximum growth rates $(\omega_{iN})_{max}$ computed by Lessen *et al.* The maxima occur at the $\beta = \alpha/n$ values indicated for the two cases. The last column provides a partial check on the validity of the asymptotic theory and the numerical computations.

q	$(\omega_{iAS})_{max}$	$(\omega_{iN})_{max}$	ω_{rAS}	ω_{rN}	β_{AS}	β_N
0·8	0·371	0·378	−1·171	−1·112	0·505	0·538
1·0	0·356	0·348	−1·552	−1·539	0·602	0·613
1·2	0·257	0·238	−1·998	−2·139	0·679	0·638
1·3	0·159	0·152	−2·281	−2·514	0·702	0·638

TABLE 3. Comparison of the results of the asymptotic theory for the maximum growth rate at fixed q for $n = 4$ with numerical solutions computed by the method of §3. Corresponding values of the real frequencies are listed, together with the normalized wavenumber at which $(\omega_i)_{max}$ occurs. $(\ )_{AS} \longleftrightarrow$ asymptotic theory, $(\ )_N \longleftrightarrow$ numerical solutions.

The results calculated from the asymptotic formulae (4.39) and (4.40) may be compared with numerical data for the principal mode accumulated by Lessen *et al.* (1974) and by ourselves. Table 2 compares the maximum growth rate found by Lessen *et al.* (1974) with the asymptotic results. In computing the asymptotic results, we used the same q and α (or β) used by Lessen *et al.* (The search for the maximum ω_i over all α and q gives slightly different values when the asymptotic theory is used – this will be discussed below.) The columns marked ω_{iAS} give the maximum growth rate (over α) predicted by the asymptotic theory for the given values of n and q, while ω_{iN} give the numerical result obtained by Lessen *et al.* (1974).

Similar labels are used to indicate the wavenumber ratios $\beta = \alpha/n$ at which these maxima are attained. The last column gives $(\omega_{iN}-\omega_{iAS})n^2$: the numbers in this column provide a good indication that our solution is in fact an asymptotic representation accurate to $O(n^{-\frac{3}{2}})$ and an indication of the accuracy of the calculations.

Table 3 compares the maximum (over α) value of ω_i for $n = 4$ and various q with numerical solutions that we have computed by the method of section 3. The corresponding values of ω_r are also compared. The agreement, like that shown in table 2, is good.

We have also used the asymptotic theory to compute the overall maximum growth rate, maximized over β and q holding n fixed. These results are displayed in table 4.

The variations of ω_i and ω_r with normalized wavenumber β for $n = 4$ and several values of q are displayed in figures 2 and 3. For clarity, results that we have calculated from the differential equation are shown as solid lines, and results from the asymptotic theory are shown as broken lines. Our numerical results (solid lines) for $n = 4$ for the primary mode are also presented in figures 2 and 3. The tick on each

n	q	β	$(\omega_i)_{max}$	ω_r/n
3	0·845	0·536	0·346	−0·284
4	0·856	0·532	0·373	−0·320
5	0·865	0·531	0·390	−0·336
6	0·862	0·531	0·400	−0·343
∞	0·870	0·532	0·459	−0·363

TABLE 4. The overall maximum growth rates, maximized over β and q for fixed n, and corresponding normalized frequencies, according to the asymptotic theory

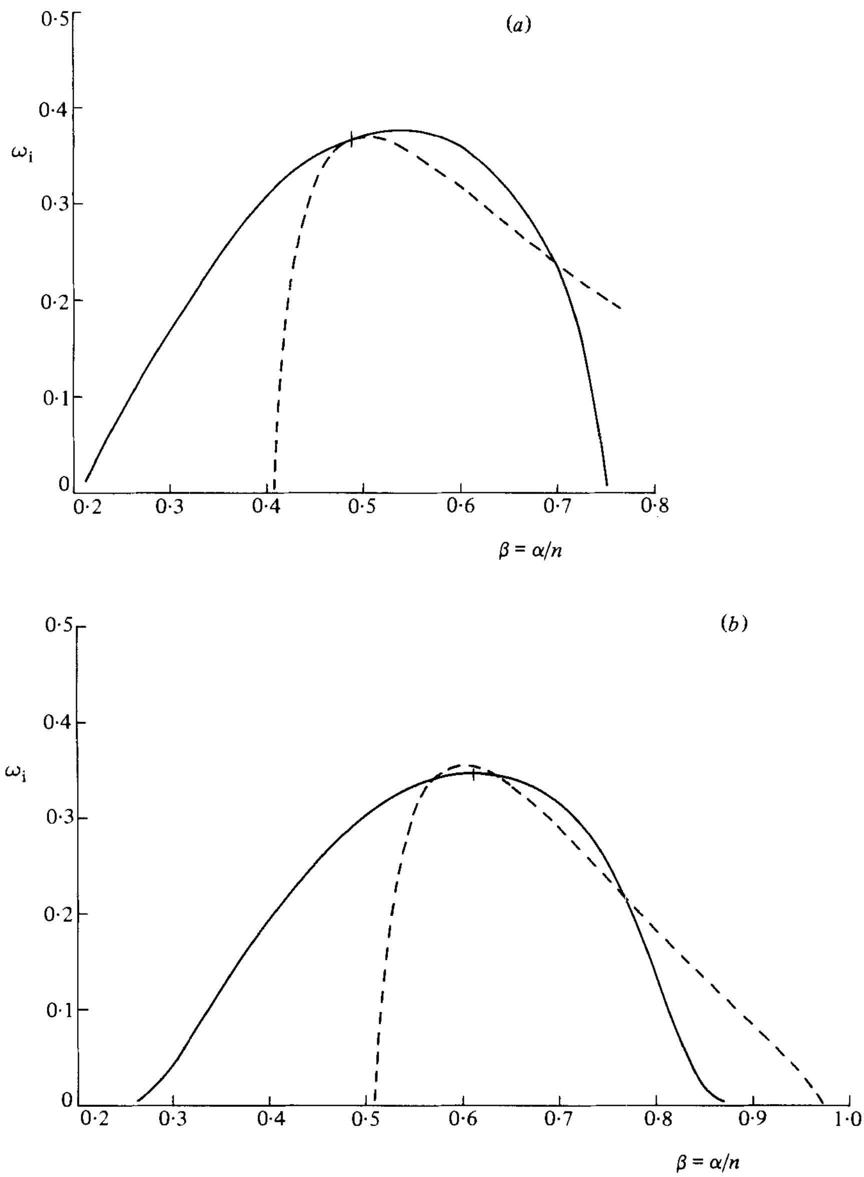

FIGURE 2(a, b). For caption see facing page.

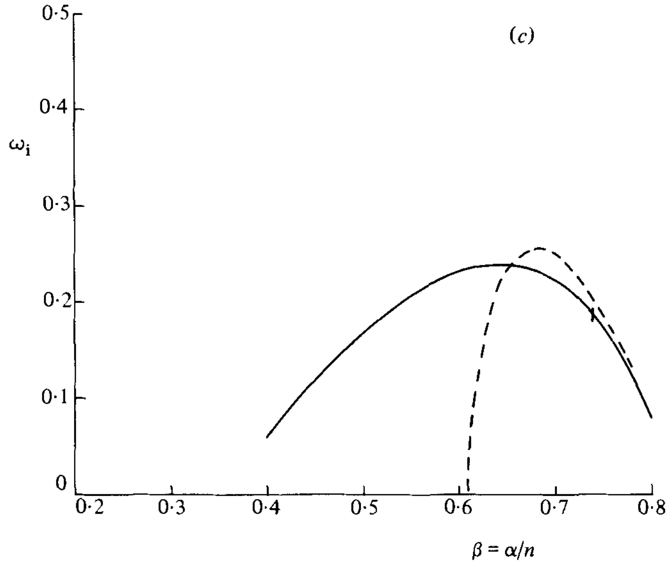

FIGURE 2. Growth rates for $n = 4$ as functions of normalized wavenumber for the trailing vortex. ——, results computed numerically by the method of §3; ----, asymptotic theory of §4. The tick marks on the solid curves mark the wavenumber for which $\gamma_r(0) = 0$. (*a*) $q = 0{\cdot}8$; (*b*) $q = 1{\cdot}0$; (*c*) $q = 1{\cdot}2$.

curve of $\omega_i(\beta)$ in figure 2 marks the value of $\beta = \alpha/n$, say β_1, at which $\gamma_r = 0$ at $r = 0$. For wavenumbers on the curve with $\beta < \beta_1$, there is only one point, $r = r_{c1}$, at which $\gamma_r(r_c) = 0$. As β increases, r_{c1} increases, and, when β reaches β_1, γ_r vanishes at a second point, $r_{c2} = 0$. As β increases beyond β_1, both r_{c1} and r_{c2} increase. We also display in figure 4 the results found by Lessen *et al.* (1974) for other values of n. The agreement between the numerical results and the asymptotic theory is good near the wavenumber corresponding to the maximum growth rate, but deteriorates rather quickly on either side. The present asymptotic analysis fails near $\beta = \frac{1}{2}q$, and also near $\beta q = 1$. Near $\beta = \frac{1}{2}q$, the failure can be traced to the approximation of γ by a quadratic ($r_0 = 0$ in (4.14) in this case, and $\Lambda_2 = 0$), and in the case $\beta q \to 1$ because $b_0 \to 0$. The expansion may be made uniform by attending to these causes of failure. The analysis required will be presented elsewhere; one result relevant to figures 2–4, however, is worth mentioning now. We find that the effect of the nonuniformity near $\beta = \frac{1}{2}q$ is to decrease values of β corresponding to a given ω_i by an amount proportional to $n^{-\frac{2}{3}}$: for the values of n plotted in the figures the shift improves the agreement between the asymptotic theory and the numerically computed solutions.

For higher modes, the asymptotic formulae are the same to $O(n^{-\frac{1}{2}})$, except that Γ_1 has an additional multiple of $2s-1$, and the formulas are expected to accurately represent only those higher modes for which $(2s-1)n^{-\frac{1}{2}}$ is small. For relatively small n, as in Duck & Foster's (1980) results, this factor soon becomes significant, and we have therefore not attempted to compare the asymptotic theory to existing calculations of higher modes.

5. Discussion

The numerical and analytical studies carried out to date on the stability characteristics of the trailing vortex may be summarized as follows. The vortex is certainly stable to axisymmetric disturbances ($n = 0$) if $q = 0$ (since $b = 0$ and $a > 0$), and if

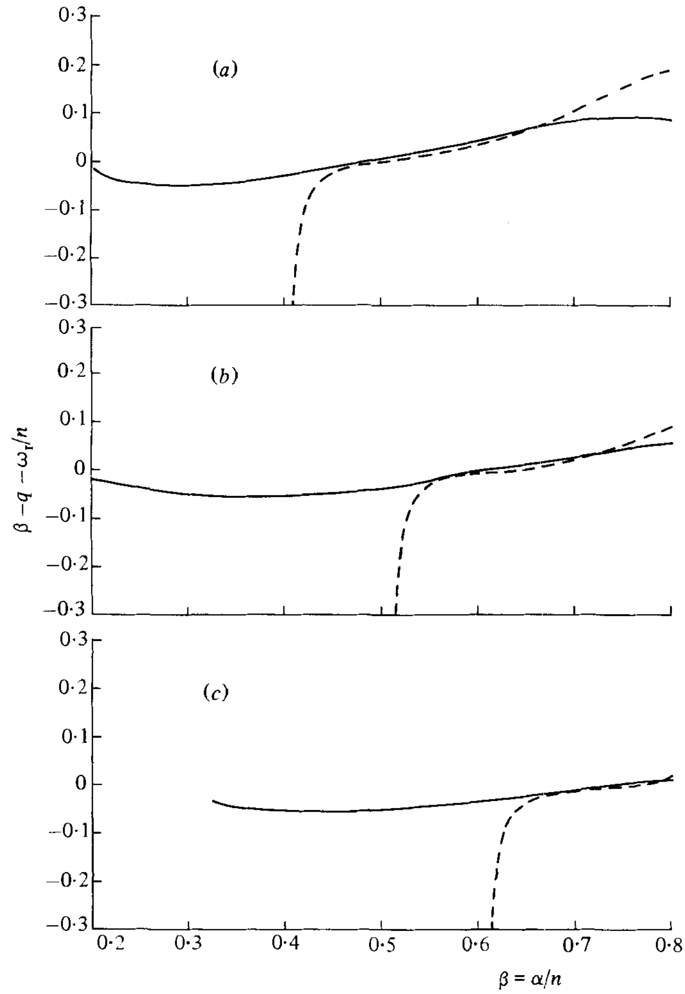

FIGURE 3. Real frequencies of unstable modes corresponding to (a), (b) and (c) of figure 2.

$q > 0{\cdot}403$ (Howard & Gupta 1962; see also (5.4) below); very likely it is stable to such disturbances for all q, but we have not seen an explicit demonstration. It is unstable to asymmetric disturbances with $n < 0$ if $q < q_0$ ($q_0 \approx 0{\cdot}08$) and stable if $q > q_0$, the most significant value of n being -1 (Lessen *et al.* 1974). Of the disturbances with $n > 0$, those with $n \gg 1$ are the easiest to study and for the most unstable of these the axial wavenumbers must satisfy

$$\tfrac{1}{2}q < \frac{\alpha}{n} < \frac{1}{q}. \tag{5.1}$$

This class of disturbance can however only exist if $q < \sqrt{2}$, for if $q > \sqrt{2}$ this inequality leads to a contradiction. Hence we may certainly claim that a sufficient condition for the instability of this swirling flow is

$$q < \sqrt{2}. \tag{5.2}$$

This result may be generalized to include the general class of swirling flows considered in §2. The only general condition for instability so far proposed is that

$$\alpha^2 r^{-3} D(r^2 V^2) + 2\alpha n r^{-2} V DW - \tfrac{1}{4}[\alpha DW - nD(V/r)]^2 < 0 \tag{5.3}$$

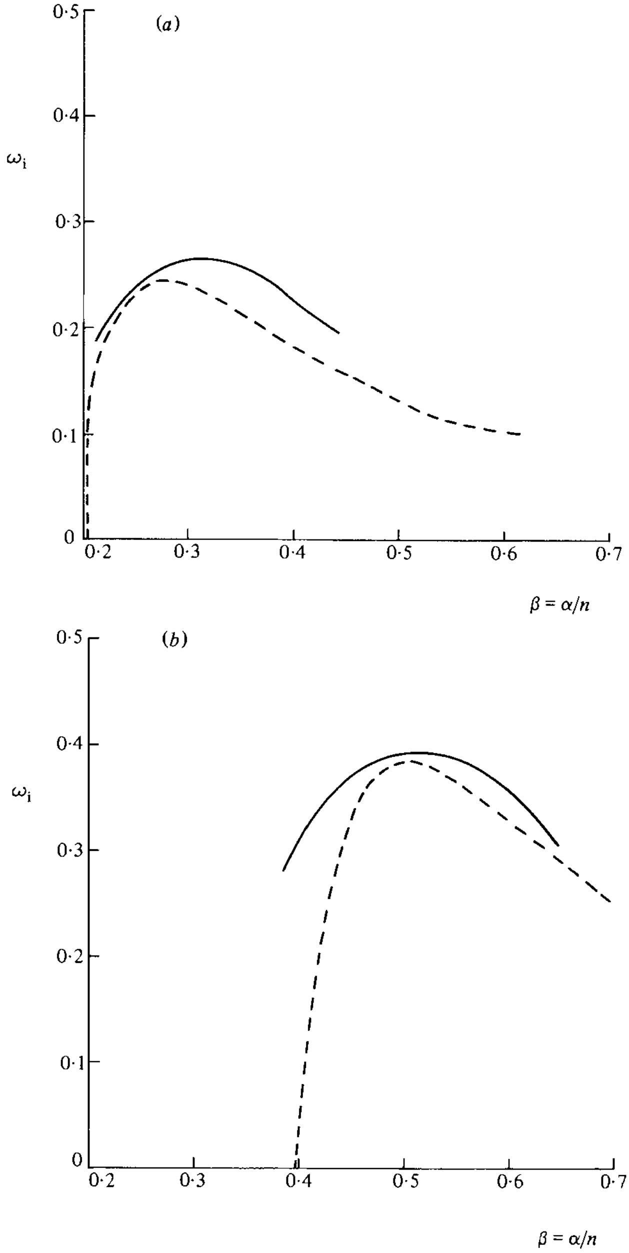

FIGURE 4. For caption see p. 354.

for some $r > 0$ (Howard & Gupta 1962) and this condition is *necessary*. It is useful for deciding *stability* when α, n are given. For example, we have immediately that if the flow is unstable to symmetric disturbances ($n = 0$) then

$$r^{-3}\frac{d}{dr}(r^2V^2) < \frac{1}{4}\left(\frac{dW}{dr}\right)^2, \tag{5.4}$$

but the condition is not sufficient, and, as the authors point out, (5.3) is satisfied for any V, W and $n \neq 0$ provided α is small enough. No *sufficient* conditions for the

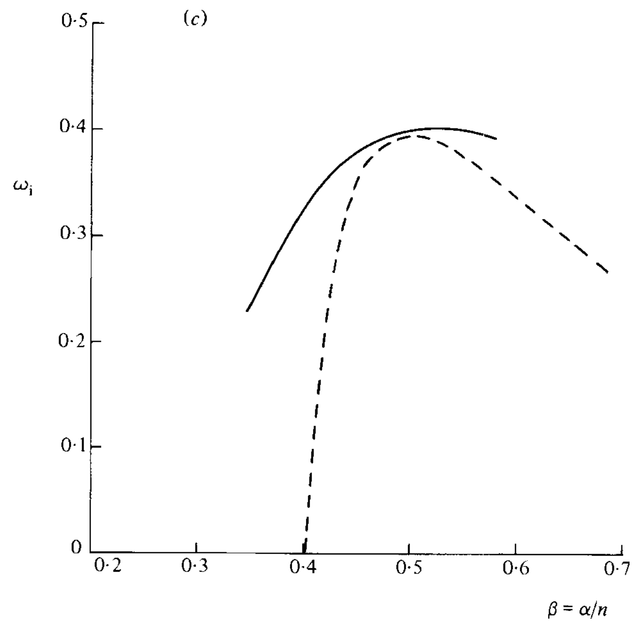

FIGURE 4. Solid lines show growth rates computed numerically by Lessen *et al.* (1974), dashed lines are from the asymptotic theory. (*a*) $q = 0{\cdot}4$, $n = 4$; (*b*) 0·8, 5; (*c*) 0·8, 6.

instability of a general columnar vortex are available in the literature. Ludwieg (1961) has proposed a necessary and sufficient condition for stability, namely, the flow is stable if and only if

$$\frac{-D(rV)\,[D(V/r)]^2}{D(V/r)-2V/3r^2} > (DW)^2. \tag{5.5}$$

This condition emerged from his study of flow between concentric cylinders, but for the TV flows this implies that they are always stable. This condition is too severe. It was derived for the special case of a narrow annular gap, and its application to other geometries has no rational basis.

We now proceed to formulate a sufficient condition for a columnar vortex of finite or infinite extent to be unstable. We first take α, n large and look for a stationary value of γ. This occurs at a value of r that satisfies

$$\frac{DW}{D(V/r)} = \frac{n}{\alpha}. \tag{5.6}$$

The growth rate of this disturbance satisfies

$$\omega_{\mathrm{i}}^2 = -\frac{2\alpha V}{r^2}S\left[\alpha\left(r\frac{dV}{dr}+V\right)+n\frac{dW}{dr}\right], \tag{5.7}$$

apart from contributions that tend to zero as $n \to \infty$ and provided that the right-hand side is positive. On making use of (5.6) we deduce that

$$\omega_{\mathrm{i}}^2 \to \frac{2V[rDV-V][V^2/r^2-(DV)^2-(DW)^2]}{(rDV-V)^2+r^2(DW)^2} = \omega_{\infty\mathrm{i}}^2(r). \tag{5.8}$$

Hence if the maximum growth rate for any disturbance is σ_{M} then

$$\sigma_{\mathrm{M}} \geqslant \max_r \omega_{\infty\mathrm{i}}; \tag{5.9}$$

the equality sign appears to hold for TV flows if $q < \surd 2$, but not generally (see below). An upper bound for $\sigma_{\max}$ has been provided by Howard & Gupta (1962), namely

$$\sigma_{\mathrm{M}}^2 \leqslant \max \frac{r^2}{1+\beta^2 r^2}\left\{\tfrac{1}{4}\beta^2 (DW)^2 - \frac{\beta^2[D(r^2V^2)]}{r^3} - \frac{\beta\, DW\, D(r^3V)}{2r^4} + \left[D\left(\frac{V}{r}\right)\right]^2\right\}, \quad (5.10)$$

where the maximum is taken over all β and over all positive r. The right-hand side of (5.10) is greater than $\omega_{\infty\mathrm{i}}^2$ by

$$\frac{3r^2(DW)^2(rDV-V)^2}{4[(rDV-V)^2+[rDW]^2]}, \quad (5.11)$$

when β is defined by (5.6): this serves as a consistency check of (5.9).

The result (5.9) provides us with our sufficient condition for the instability of columnar vortices, which may be stated as follows. Let V, W be smooth functions of r, but $\Gamma = Vr$ be the circulation and let $\Omega = V/r$ be the angular velocity. Then the flow must be unstable if

$$V\frac{d\Omega}{dr}\left[\frac{d\Gamma}{dr}\frac{d\Omega}{dr}+\left(\frac{dW}{dr}\right)^2\right] < 0 \quad (5.12)$$

at any point of the flow field.

This condition is not necessary for instability. An example is provided by swirling Poiseuille flow (Maslowe 1974; Maslowe & Stewartson 1982). Here $V = r$, $W = \epsilon(1-r^2)$ for $0 \leqslant r \leqslant 1$, and this flow is unstable even though (5.12) is violated. When n is large there is a class of unstable wall modes with critical levels at distances $O(1/n)$ from the pipe wall at $r = 1$. It seems clear that these modes are consequences of the flow properties $V \neq 0$ and $dW/dr \neq 0$ at $r = 1$. They also show that the maximum growth rate does not occur in the limit $n \rightarrow \infty$ if ϵ is large enough: further, in (5.9), the inequality sign is required since $\omega_{\infty\mathrm{i}} = 0$.

Even for unbounded flows the criterion (5.12) is not necessary. The comparison between the numerical studies reported here and the asymptotic theory suggest that the maximum growth rate is achieved in the limit $n \rightarrow \infty$, and for large enough n the growth rate is a monotonic function of n for fixed β. This theory, however, only applies when (5.1) is satisfied, and we know that for finite n the range of values of β for which $\omega_{\mathrm{i}} > 0$ extends below $\frac{1}{2}q$. There is no evidence from the numerical studies that the range of β for unstable modes extends beyond $\beta = 1/q$, and there are no eigenvalues of (2.1) if $\beta q = 1$ and $q > \frac{1}{2}$. For then $b = 0$, $a > 0$ and they are excluded by a well-known property of Rayleigh's equation.

An extension of the theory to include values of $\beta < \frac{1}{2}q$ is clearly needed, and this will be the subject of a later paper. More germane to the present discussion is that a preliminary numerical study of the unstable modes near the likely position of the lower neutral mode has been carried out for $n = 3$, 4, 5, and very weakly unstable modes have been found in the range $1{\cdot}42 \leqslant q \leqslant 1{\cdot}58$. The marginal character of the instability may be gauged by noting that at $n = 4$, $q = 1{\cdot}50$ the most-unstable mode occurs at $\alpha = 2{\cdot}32$ and then $\omega = -3{\cdot}658 + 0{\cdot}00152i$. The value of $\gamma_{\mathrm{r}}(0) \approx -0{\cdot}02$ and the critical level is at $r \approx 0{\cdot}2$. Further studies of these modes are needed before a definite conclusion can be drawn about the maximum value of q for which unstable modes can occur, but there is some evidence that, as q decreases from infinity, the first mode to become unstable is $n = 1$. In their numerical study of viscous modes Lessen & Paillet (1974) found that the critical Reynolds number is least for the $n = 1$ mode, and their conclusions are confirmed by Stewartson (1982), using an extension of the present analysis.

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